Lessons 016 - 017
Continuous Random Variables

Friday, October 13

102
101
100
54
55
56
92
91
90
64
65
66
82
81
80
74
75
76

PMFs with Continuous Random Variables

- Recall that a pmf, $p(x)$, is a function that assigns a probability value to **every** possible realization in the sample space.
- A continuous random variable is defined by having an **uncountable** number of possible values to take on.
- If a positive probability were assigned to an uncountable number of events, we would be assigning infinite probability.
    - We would also be unable to work with these quantities at all.

CDFs as the Solution

- Recall that the CDF was defined as $F(x) = P(X \leq x)$.
- This was given by a summation of the pmf over possible values.

$$F(k) = \sum_{x=-\infty}^{k} p(x)$$

- This is still well defined for continuous random variables.

Continuous Random Variable: CDFs

- If X is a continuous random variable, we define the CDF as:

$$F(x) = P(X \leq x)$$

- This is exactly analogous to the discrete CDF.

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

A continuous random variable has CDF $F(x) = 1 - e^{-x}$. What is the probability that X is less than or equal to 1?

$1 - e^{-1}$

0%

$\int_{-\infty}^{1} (1 - e^{-x}) dx$

0%

e^{-1}

0%

0

0%

A continuous random variable has CDF $F(x) = 1 - e^{-x}$. What is the probability that X is exactly equal to 1?

$1 - e^{-1}$

0%

$\int_{-\infty}^{1} (1 - e^{-x}) dx$

0%

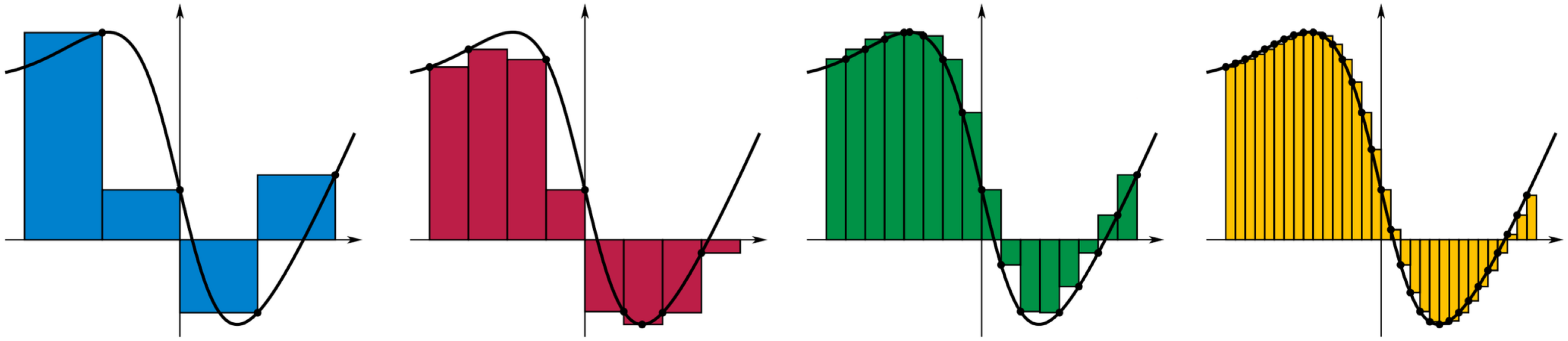
e^{-1}

0%

1

0%

The correct answer is 0.
Remember, the probability that a continuous random variable takes on any **specific** value is always 0.



The Probability Density Function

- We can use the Fundamental Theorem of Calculus

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

- This gives us that $f(x) = \frac{d}{dx}F(x)$.

- We call $f(x)$ a **probability density function.**

The Probability Density Function: Continued

- The PDF characterizes a continuous distribution.
  - It can be plotted similar to a PMF.
- The PDF does **not** give probabilities directly, instead we integrate it for probabilities.
- We must have $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1.$

A continuous random variable has CDF $F(x) = 1 - e^{-x}$. What is the probability density function of X?

$1 - e^{-x}$

0%

$1 + e^{-x}$

0%

e^{-x}

0%

e^{x}

0%

Computing Probabilities

- Given a PDF, $f(x)$, we say that

$$P(X \in [a,b]) = \int_a^b f(x)dx$$

- Can also write this as $F(b) - F(a)$.
- Recall that $P(X = c) = 0$ for every c.

Computing Probabilities

- Given a PDF, $f(x)$, we say that

$$P(X \geq a) = \int_a^\infty f(x)dx = 1 - F(a)$$

- Since $P(X = a) = 0$, this is the same as $P(X > a)$.

The PDF of X is given by $f(x) = 1$ for $0 \leq x \leq 1$ and 0 elsewhere. What is the probability that X is between $\frac{1}{4}$ and $\frac{3}{4}$?

1

0%

$f(\frac{3}{4}) - f(\frac{1}{4})$

0%

0

0%

$\frac{1}{2}$

0%

The PDF of X is given by $2e^{-2x}$, for $X \geq 0$. What is the probability that $X = 3$?

$2e^{-6}$

0%

$\int_0^3 2e^{-2x}\,dx$

0%

$\int_{-\infty}^3 2e^{-2x}\,dx$

0%

0

0%

Suppose that X has a PDF of $f(x) = \frac{1}{3}x^2$ on $[-1, 2]$ and 0 elsewhere. What is the CDF of X?

Do not forget the bounds of integration! I did (and as a result got the wrong answer :(, the correct answer is shown).

0%

$F(x) = \frac{2}{3}x$

0%

$F(x) = x^3$

0%

$F(x) = \frac{x^3}{9}$

0%

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

$$= \int_{-1}^{x} \frac{1}{3}x^2$$

$$= \frac{t^3}{9}\Bigg|_{t=-1}^{x}$$

$$= \frac{x^3}{9} - \frac{(-1)^3}{9}$$

$$= \frac{x^3+1}{9}$$

A continuous random variable has CDF $F(x) = 1 - e^{-x}$. What is the probability that $X \geq 4$?

$1 - e^{-4}$

0%

$1 - e^{4}$

0%

e^{-4}

0%

e^{4}

0%

Discrete versus Continuous Random Variables

- Summations become integrals.

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

$$E[h(X)] = \int_{-\infty}^{\infty} h(x)f(x)dx$$

$$\text{var}(X) = E[(X - E[X])^2] = \int_{-\infty}^{\infty} (x - E[X])^2 f(x)dx$$

Suppose that a random variable has desnity $f(x) = \frac{1}{2}$ on $[0, 2]$ and 0 elsewhere. What is $E[X]$?

1	0%
$\frac{1}{2}$	0%
2	0%
0	0%

The Continuous Uniform Distribution

- If X is equally likely to take on any value on an interval $[a, b]$, it has a **uniform distribution.**
- $X \sim \text{Unif}(a, b)$
- $f(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$
- $E[X] = \frac{a+b}{2}$ and $\text{var}(X) = \frac{(b-a)^2}{12}$

What is the CDF of X if $X \sim \mathrm{Unif}(a, b)$? (Only on $[a, b]$)

$F(x) = \frac{1}{b-a}$.

0%

$F(x) = \frac{x-a}{b-a}$

0%

$F(x) = \frac{x}{b-a}$

0%

$F(x) = \frac{x}{b} - \frac{x}{a}$.

0%

Percentiles Revisited

- Recall that the $100p$th percentile of a distribution divides data so that $100p\,\%$ of the data are below and $100(1-p)\,\%$ are above.
- Define $\eta(p)$ to be the $100p$th percentile.
  - $\eta(0.5)$ is the median.
  - $\eta(0.99)$ is the 99th percentile.
  - $\eta(0.25)$ is Q1 and $\eta(0.75)$ is Q3.

Computing Percentiles

- With continuous distributions, can often solve for $\eta(p)$.

$$F(\eta(p)) = p \implies p = \int_{-\infty}^{\eta(p)} f(x)dx$$

Suppose that X is a random variable, defined on $[0, 10]$ with an unknown distribution function. If $\eta(0.25) = 2$, $\eta(0.5) = 5$, and $\eta(0.75) = 6$. Which probability is the highest?

$P(X \leq 2)$

0%

$P(2 \leq X \leq 5)$

0%

$P(5 \leq X \leq 6)$

0%

$P(6 \leq X \leq 10)$

0%

All of these probabilities are the same.

0%

There is not enough information provided.

0%